THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XI. MARCH, 1904. No. 3.

SPHERICAL GEOMETRY.

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LECTURE III. THE CONCEPT OF ORDER; ANGLES AND TRIANGLES.

Theorem 6. *All the lines which pass through the same point O pass also through a common point O'.*

Let the two lines a and b which pass through the point O meet again in some point O'. Let c be any other line which passes through the point O. Then c must cut each of the lines a and b in another point—say Q and R respectively. The pairs of points O and O', O and Q, O and R do not uniquely determine a line: for we have by the above hypothesis two lines through each pair of points. Now Axiom I states that there exists only one point which taken with the point O will not uniquely determine a line. Hence the points O' and Q and R must coincide and the theorem is proved.

Theorem 7. *If in a motion one point of the surface remains stationary, that other point which taken with it will not uniquely determine a line remains also stationary.*

If O be the point which remains fixed, any direction a issuing from O will be carried into a direction a' which issues from O. Hence if b and c are any two lines passing through O they will be carried into two lines b' and c' which likewise pass through O. But by the foregoing theorem b and c intersect in O' and b' and c' also intersect in O'. Hence O' must be carried into O'—that is, it remains unchanged in position.

Two points such as O and O' through which an indefinitely great number of lines may be passed are called *antipodal* points. The segment of a line intercepted between the two points is called a *semi-line*.

Theorem 8. *All semi-lines are congruent and in particular are congruent to the remaining portions of the lines of which they are severally the parts.*

That all semi-lines intercepted between the same points are congruent is evident from the fact that, O and O' remaining stationary, any one direction issuing from O may be carried into any other. To compare semi-lines intercepted between different pairs of antipodal points it is only necessary to perform a motion which carries one of the points of one pair into one of the points of the other pair. The number of steps to follow out in the reasoning is not large and they will be left to the reader to fill in.

Definition of angle: *An angle is the geometric figure formed by two directions issuing from a common point called the vertex of the angle.* The two directions, apart from any particular segments lying along them, are known as the sides of the angle.

It should be noted that when sufficiently produced the sides of an angle meet; that the angle must not be regarded as the portion of the spherical surface intercepted between its sides but merely as the geometric figure consisting of the sides which are of limited but undefined extent and of the vertex.

Theorem 9. *An angle is congruert to itself and is not congruent to any angle which has one side and the vertex in common with it but the other side different.*

The proof is like the proof of the corresponding theorem for segments. In fact there is a great similarity in the theorems concerning angles and segments. For example we set it down as an axiom that the points of a line are arranged in a natural order so that the words "follow," "precede," "lie between" are applicable, and we found this axiom very useful. It is intuitively obvious that the same is true of angles about a common vertex, or, to speak more precisely, of the directions issuing from a given point. We might, therefore, either postulate this property of angles or prove it, if possible, by means of the corresponding property of lines which we have already postulated or we mav be able to get along without it by using indirectly the properties of the arrangement of points in lines. Which of these courses we choose to pursue depends to a considerable extent on our point of view.

As the idea of *order* is of the utmost importance in many diverse branches of mathematics and may often be taken as fundamental, the following discussion will be not inappropriate. Consider the digits

	0	1	2	3	4	5	6	7	8	9
and	5	6	7	8	9	0	1	2	3	4
and	0	3	6	9	0	5	8	1	4	7
and	0	7	4	1	8	5	2	9	6	3.

Of these four methods of arrangement the first will unquestionably be regarded as the only natural arrangement. The second is not essentially different from the first. For if the series of elements be considered to form a closed cycle so that the last written leads back to the first, the application of the words "follow," "precede," and "lie between" is not impaired in the slightest. In each set 7 falls

between 6 and 8. On the other hand the last two arrangements are different from the former and are different from each other. The order has been essentially disturbed and has been replaced by a different order which for some problems in the theory of numbers may be more convenient but which nobody would take to be the natural order. If we look a little more carefully we see that the last arrangement does not differ so radically from the third as at first appears to be the case. The order in the two has merely been reversed. The words "lie between" have the same validity in both but the meaning of "follow" and "precede" has been interchanged.

These observations lead to the idea that when a set of elements are arranged in a closed series, two arrangements in which the words "lie between," "follow," and "precede" have the same applicability may be called the same arrangement; two arrangements in which "lie between" preserves its characteristics, and "follow" and "precede" are interchanged may be called opposite; and two arrangements in which all three of the words change their significance when applied to the individual elements may be considered as essentially distinct. If there be given two sets of elements

0 1 2 3 4 5 6 7 8 9 0

and A B C D E F G H I J A

which may be paired off in such a way that to each element of either set there corresponds one element of the other and only one and if each set possesses a definite arrangement of its members and if furthermore the words "follow," "precede," and "lie between" have the same significance when applied to the different members of each pair then the two sets of elements may be said to have the same arrangement.* Such is evidently the case in the example cited above.

When there is given a set of elements such as

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which do not possess any natural order or of which the natural order is not known, an order may be attributed to them arbitrarily by pairing them off against a set of elements which are arranged in order and by bodily transferring this order to them. Such a proceedure is often useful but it must be remembered that no real or natural order has thereby been obtained among the elements considered unless it can be shown that every such method of establishing an order leads to the same final result. For instance, let P be any point on the spherical surface. Describe any line which does not pass through the point P. Each di-

*The fact each of two sets of elements must possess its own independent characteristics if any valuable conclusions are to be drawn from establishing a correspondence between the sets has been overlooked by so eminent a mathematician as Hilbert. See my paper on The So-called Foundations of Geometry, *Archiv der Mathematik und Physik*, Vol. VI, pp. 104-122, 1903. It must however be evident that if the elements of a set B have no other properties than those derived from a one to one correspondence with the elements of a set A, then the set B is trivial and any thoorem obtained concerning its elements is really a theorem concerning the elements of the original set A. For this reason logicians are careful to compare not only the members of two sets but the operations and conceptions connected with them.

rection a issuing from the point P will cut this line in one point, say in A. The points A are arranged in a definite order upon the line and we may say that the directions a are arranged in the same order. We may say that a direction a' follows the direction a when the corresponding point A follows the point A. The order which is thus attributed to the directions a is purely relative to and dependent on the order of the points on the particular line which has been chosen. Had some other line been fixed upon and the same process carried out there might conceivably have resulted a wholly different arrangement of the directions issuing from the point P. And until it has been proved that no matter what line was selected the same arrangement of the directions must result, we are in no position to affirm that the order found is at all a natural order.

To bring out even more clearly some of the ideas connected with order consider the series of positive integers and the series of all positive rational numbers. Each of these series possesses an obvious natural order; yet by rearranging the latter into an order wholly artificial it becomes possible to pair off the elements of the two series in a one to one manner. Any rational number may be written in the form p/q where p and q have no common factor. Arrange the rational numbers so that those in which the sum of numerator and denominator is smaller precede those in which the sum of numerator and denominator is greater and in case the sums are the same place the numbers which have the smaller numerator first. There results the series

$$1 \quad 2 \quad \tfrac{1}{2} \quad 3 \quad \tfrac{1}{3} \quad 4 \quad \tfrac{1}{4} \quad \tfrac{2}{3} \quad \tfrac{3}{2} \quad 5 \quad \tfrac{1}{5} \quad 6 \quad \tfrac{1}{6} \quad \tfrac{2}{5} \quad \tfrac{3}{4} \quad \tfrac{4}{3} \quad \tfrac{5}{2} \quad 7 \quad \ldots\ldots\ldots$$

which contains once and only once every rational number and which may be put into one to one correspondence with the series of integers

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad \ldots\ldots\ldots$$

Thus by sacrificing the natural order we may set up a one to one correspondence between these two series. It has been shown by somewhat complicated analysis that the points upon the segment of a line may be paired off against the points in a square. In this manner the points in a square may be considered to be ordered into a simple series in which the words "lie between," "follow" and "precede" are applicable. As however this order is wholly artificial and may be differently established in a great number of ways is is of very limited value.

Considerations like these shed a great light on the magnitude and significance of the assumptions at the bottom of the third axiom which states that the points of a line are arranged in a natural order. The fact of the arrangement alone is not of great import because almost any set of elements may be assigned an order. The word *natural* is what gives the axiom its significance. It states that the points of a line are distributed in an order which cannot be mistaken for nor confused with any other. If there were a number of equally natural orders which could not be at all times readily distinguished from one another the whole theory of segments would become unsettled. It also becomes clear that before

speaking of a natural order among the directions issuing from a point so many difficulties are to be met that a development of the theory of angle along lines similar to those pursued in discussing segments had best not be attempted any earlier than necessary.

Definition of (proper) triangle: *A (proper) triangle is the geometric figure composed of three segments each of which is less than a semi-line drawn so as to connect in pairs three points.* The parts of the triangle are its three sides, and its three angles which are the angles formed by the directions in which the segments leave the vertices. A segment less than a semi-line may be called a proper segment.

Theorem 10. *If two sides and the angle formed by them in one triangle are congruent to two sides and the angle formed by them in another triangle, the triangles are congruent.*

As two angles are congruent the triangles may be moved so that these angles coincide. The adjacent sides take the same directions and as they are congruent they must coincide throughout. The third sides will therefore lie on the same line and between the same points of the line. Of the two segments which satisfy these conditions one is greater than a semi-line and cannot form the side of a proper triangle. Hence the other segment must be the third side of each triangle and the triangles coincide throughout.

Theorem 11. *If a side and two adjacent angles of one triangle are congruent to a side and two adjacent angles of a second triangle, the triangles are congruent.*

Theorem 9 is needed in the proof—which is left to the reader. A similar proof may be given for the following:

Theorem 12. *If two angles are congruent their vertical angles are also congruent.*

Theorem 13. *If three directions a, b, c radiate from a point and three directions a', b', c' from the same or a different point and if furthermore the congruences* $\measuredangle ab \equiv \measuredangle a'b'$ and $\measuredangle ac \equiv \measuredangle a'c'$ *are fulfilled, then* $\measuredangle bc \equiv \measuredangle b'c'$.

ON SOME SPECIAL ARITHMETIC CONGRUENCES.

By H. S. VANDIVER, Bala, Pa.

If we take the well known relation

$$\left(\frac{p-1}{2}\,!\right)^2 \equiv (-1)^{(p+1)/2} (\bmod p),$$

where p is a prime, and suppose that $p \equiv 3 (\bmod 4)$, then

$$\left(\frac{p-1}{2}\,!\right)^2 \equiv 1 (\bmod p) \ldots\ldots (1),$$